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Computational Aspects of Asymptotic Matching in the Restricted Three-Body Problem

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THE results of asymptotic matching in the restricted three-body problem yield an asymptotic approximation to the perturbed hyperbola about the moon in an Earth-to-moon trajectory.[1–4] The approximation is given in terms of the initial conditions, small variations in the initial conditions and definite integrals of bounded continuous functions. The definite integrals, however, involve the bounded difference of two quantities which themselves become unbounded at the upper limit of integration. The purpose of this Note is to describe the technique used to compute these definite integrals.

Let $\boldsymbol{r}(t, \mu)$ be the solution of the restricted three-body problem

$$\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}) + \mu \boldsymbol{f}(\boldsymbol{r}, t), \quad \mu \ll 1$$

$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0(t_0) + \delta \boldsymbol{r}_0, \quad \dot{\boldsymbol{r}}(t_0) = \dot{\boldsymbol{r}}(t_0) + \delta \dot{\boldsymbol{r}}_0 \tag{1}$$

where

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{r}/|\boldsymbol{r}|^3$$

$$\boldsymbol{f}(\boldsymbol{r}, t) = -\left[\frac{\boldsymbol{r} - \boldsymbol{r}_m(t)}{|\boldsymbol{r} - \boldsymbol{r}_m(t)|^3} + \frac{\boldsymbol{r}_m(t)}{|\boldsymbol{r}_m(t)|^3} - \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right]$$

and where $\boldsymbol{r}_m(t)$, the moon's position at time t, satisfies

$$\ddot{\boldsymbol{r}}_m = \boldsymbol{F}(\boldsymbol{r}_m) \tag{2}$$

with the initial conditions $\boldsymbol{r}_m(t_0)$ and $\dot{\boldsymbol{r}}_m(t_0)$ chosen so that $\boldsymbol{r}_m(t) \neq 0$ for $t \geq t_0$. Let $\boldsymbol{r}_0(t)$ be a solution of the two-body problem

$$\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}(\boldsymbol{r}_0) \tag{3}$$

with the initial conditions $\boldsymbol{r}_0(t_0)$ and $\dot{\boldsymbol{r}}_0(t_0)$ chosen so that $r_0(t) \neq 0$ and so that at $t_1 > t_0$

$$\boldsymbol{r}_0(t_1) = \boldsymbol{r}_m(t_1)$$

and

$$\boldsymbol{V}_1 \equiv \dot{\boldsymbol{r}}_0(t_1) - \dot{\boldsymbol{r}}_m(t_1) \neq 0$$

i.e., the initial conditions $\boldsymbol{r}_0(t_0)$ and $\dot{\boldsymbol{r}}_0(t_0)$ are chosen so that the conics $\boldsymbol{r}_0$ and $\boldsymbol{r}_m(t)$ intersect at a time $t_1 > t_0$ with nonzero relative velocity. It is assumed that t_1 is the first such time; i.e., it is assumed that $\boldsymbol{r}_0(t) \neq \boldsymbol{r}_m(t)$ for $t_0 \leq t < t_1$. (Note that in moon-to-moon orbits it is assumed that $\boldsymbol{r}_0(t_0) = \boldsymbol{r}_m(t_0)$ and that $\boldsymbol{r}_0(t_1) = \boldsymbol{r}_m(t_1)$; however, moon-to-moon orbits which are treated in Ref. 1 will not be considered here.)

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Under the above assumptions, the method of asymptotic matching[1–4] can be applied to the restricted three-body problem in order to obtain an asymptotic approximation to the solution $\boldsymbol{r}(t, \mu)$ which, in the case of first-order matching, has been shown to be uniformly valid to $0(\mu^2)$ during one moon passage; i.e., over a time interval $[t_0, t_1 + 0(\mu)]$; cf. Refs. 4 and 5.

In the method of asymptotic matching in the restricted three-body problem one first expands the solution $\boldsymbol{r}(t, \mu)$ of Eq. (1) relative to the Earth in an asymptotic series (called the outer expansion)

$$\boldsymbol{r}(t, \mu) \simeq \boldsymbol{r}_0(t) + \mu \boldsymbol{\rho}_1(t) + \mu^2 \boldsymbol{\rho}_2(t) + \cdots$$

where $\boldsymbol{r}_0(t)$ is the solution of the two-body problem (3) and it is shown in Ref. (1) that the first-order perturbation

$$\boldsymbol{\rho}_1(t) = \boldsymbol{\rho}_{1b}(t) + \boldsymbol{i} V_1^{-2} \ln(\tau_0/\tau)$$

where the bounded part of the first-order perturbation

$$\boldsymbol{\rho}_{1b}(t) = \Phi_{rr}(t, t_0)\delta \boldsymbol{r}_0/\mu + \Phi_{rv}(t, t_0)\delta \dot{\boldsymbol{r}}_0/\mu - \boldsymbol{i}[1 - (\tau/\tau_0)]/V_1^2$$
$$+ \int_{t_0}^{t} \{\Phi_{rv}(t, t')\boldsymbol{f}[\boldsymbol{r}_0(t'), t'] - \boldsymbol{i}(t - t')/(V_1^2 \tau'^2)\} dt'$$

with $\boldsymbol{i} = \boldsymbol{V}_1/|\boldsymbol{V}_1|$, $\tau = t_1 - t$, $\tau' = t_1 - t'$ and $\tau_0 = t_1 - t_0$. The transition matrix

$$\Phi = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

satisfies

$$\dot{\Phi}(t, \cdot) = \begin{bmatrix} 0 & I \\ d\boldsymbol{F}/d\boldsymbol{r}[\boldsymbol{r}_0(t)] & 0 \end{bmatrix} \Phi(t, \cdot) \tag{4}$$

and

$$\Phi(t, t) = I$$

where I is the identity matrix.

Next, one expands the solution

$$\boldsymbol{\rho}^*(t, \mu) = \boldsymbol{r}(t, \mu) - \boldsymbol{r}_m(t)$$

of the restricted three-body problem

$$\ddot{\boldsymbol{\rho}}^* = \mu \boldsymbol{F}(\boldsymbol{\rho}^*) - (1 - \mu)\left[\frac{\boldsymbol{\rho}^* + \boldsymbol{r}_m(t)}{|\boldsymbol{\rho}^* + \boldsymbol{r}_m(t)|^3} - \frac{\boldsymbol{r}_m(t)}{|\boldsymbol{r}_m(t)|^3}\right]$$

relative to the moon in an asymptotic series (called the inner expansion)

$$\boldsymbol{\rho}^*(t, \mu) = \boldsymbol{\rho}_0^*(t, \mu) + \boldsymbol{\rho}_1^*(t, \mu) \cdots$$

where $\boldsymbol{\rho}_0^*(t, \mu)$ satisfies the two-body equations

$$\ddot{\boldsymbol{\rho}}_0^* = \mu \boldsymbol{F}(\boldsymbol{\rho}_0^*)$$

and describes a moon centered hyperbola with parameters Δ, the distance to the asymptote of the hyperbola V_∞, the velocity at infinity on the hyperbola and t_p the time of perilune passage on the hyperbola.

The parameters of the moon centered hyperbola are determined in terms of the initial conditions and small variations in the initial conditions $\boldsymbol{r}_0(t_0)$, $\dot{\boldsymbol{r}}_0(t_0)$, $\delta \boldsymbol{r}_0$, and $\delta \dot{\boldsymbol{r}}_0$ by asymptotic matching which is carried out to first-order in Refs. 1 and 4 [including the computation of the first-order perturbation to the moon centered hyperbola $\boldsymbol{\rho}_1^*(t, \mu)$ which contributes $O(\mu^{3/2})$ terms to the first-order matching]. The asymptotic matching compares the asymptotic behavior of the inner and outer expansions in the matching region in order to determine the parameters of the moon centered hyperbola. The matching region is defined as the region where the errors in the inner and outer expansions are of the same order. This is shown in Refs. 4 and 5 to take place when $\tau = O(\mu^{1/2})$; i.e., when $|\boldsymbol{\rho}_0^*(t)| = O(\mu^{1/2})$; when the particle is at a distance of $O(\mu^{1/2})$ from the moon, the Earth-moon distance at t_1, $|\boldsymbol{r}_m(t_1)|$, being normalized to order one. The results of the asymptotic matching for the planar problem imply that

$$V_1(t_1 - t_p) = \mu \boldsymbol{i} \cdot \boldsymbol{\rho}_{1b}(t_1) + \mu/V_1^2[\ln(2V_1^3\tau_0/\mu e) - 1] + O(\mu^2)$$

$$\Delta = \mu \boldsymbol{j} \cdot \boldsymbol{\rho}_{1b}(t_1) + O(\mu^2), \quad V_\infty = V_1 + \mu \boldsymbol{i} \cdot \dot{\boldsymbol{\rho}}_{1b}(t_1) + O(\mu^2)$$

$$V_1 \Delta\alpha = \mu \boldsymbol{j} \cdot \dot{\boldsymbol{\rho}}_{1b}(t_1) + O(\mu^2)$$

where

$$e = [1 + (V_\infty^2 \Delta/\mu)^2]^{1/2}$$

$V_1 = |\boldsymbol{V}_1|$, $V_\infty = |\boldsymbol{V}_\infty|$, $\boldsymbol{i} \cdot \boldsymbol{j} = 0$, and $\Delta\alpha$ is the angle from $\boldsymbol{V}_1$ to $\boldsymbol{V}_\infty$, cf. Ref. 1 (p. 51), or Ref. 4 (p. 37). The results for the three-dimensional problem are also given in Ref. 1 (p. 51).

Thus, in order to determine the moon-centered hyperbola $\boldsymbol{\rho}_0^*(t, \mu)$ to first order, it is necessary to compute the quantities

$$\boldsymbol{\rho}_{1b}(t_1) = \Phi_{rr}(t_1, t_0)\delta\boldsymbol{r}_0/\mu + \Phi_{rv}(t_1, t_0)\delta\dot{\boldsymbol{r}}_0/\mu - \boldsymbol{i}/V_1^2 + \int_{t_0}^{t_1} \{\Phi_{rv}(t_1, t)\mathbf{f}[\boldsymbol{r}_0(t), t] - \boldsymbol{i}/(V_1^2\tau)\} dt$$

and

$$\dot{\boldsymbol{\rho}}_{1b}(t_1) = \Phi_{vr}(t_1, t_0)\delta\boldsymbol{r}_0/\mu + \Phi_{vv}(t_1, t_0)\delta\dot{\boldsymbol{r}}_0/\mu - \boldsymbol{i}/(V_1^2\tau_0) + \int_{t_0}^{t_1} \{\Phi_{vv}(t_1, t)\mathbf{f}[\boldsymbol{r}_0(t), t] - \boldsymbol{i}/(V_1^2\tau^2)\} dt$$

But this involves some difficulty since the preceding integrands are the bounded difference of two quantities which become unbounded as $t \to t_1$.

The technique used to evaluate these integrals will now be described. Briefly, one splits the integration into two parts, one from t_0 to $t_1 - \delta$ and one from $t_1 - \delta$ to t_1 with $\delta > 0$ and then expands the integrand in the integration over $[t_1 - \delta, t_1]$ in a Taylor series about t_1 in order to obtain an analytic expression for the integral from $t_1 - \delta$ to t_1 plus an error of $O(\delta^3)$ in $\boldsymbol{\rho}_{1b}(t_1)$ and an error of $O(\delta^2)$ in $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ (or an error of higher order in δ if higher order τ-terms are carried in the Taylor expansion of the integrands); viz.

$$\boldsymbol{\rho}_{1b}(t_1) = \frac{\Phi_{rr}(t_1, t_0)\delta\boldsymbol{r}_0}{\mu} + \frac{\Phi_{rv}(t_1, t_0)\delta\dot{\boldsymbol{r}}_0}{\mu} - \frac{\boldsymbol{i}}{V_1^2} - \int_{t_0}^{t_1} \Phi_{rv}(t_1, t)[\boldsymbol{r}_m(t)|\boldsymbol{r}_m(t)|^{-3} - \boldsymbol{r}_0(t)|\boldsymbol{r}_0(t)|^{-3}] dt - \int_{t_0}^{t_1-\delta} \left[\Phi_{rv}(t_1, t)\boldsymbol{\rho}_0(t)|\boldsymbol{\rho}_0(t)|^{-3} + \frac{\boldsymbol{i}}{(V_1^2\tau)}\right] dt + \begin{pmatrix} a_0 \\ -2b_0 \end{pmatrix} \frac{\delta^2}{2V_1^2} + O(\delta^3)$$

$$\dot{\boldsymbol{\rho}}_{1b}(t) = \frac{\Phi_{vr}(t_1, t_0)\delta\boldsymbol{r}_0}{\mu} + \frac{\Phi_{vv}(t_1, t_0)\delta\dot{\boldsymbol{r}}_0}{\mu} - \frac{\boldsymbol{i}}{(V_1^2\tau_0)} - \int_{t_0}^{t_1} \Phi_{vv}(t_1, t)[\boldsymbol{r}_m(t)|\boldsymbol{r}_m(t)|^{-3} - \boldsymbol{r}_0(t)|\boldsymbol{r}_0(t)|^{-3}] dt - \int_{t_0}^{t_1-\delta} \left[\Phi_{vv}(t_1, t)\boldsymbol{\rho}_0(t)|\boldsymbol{\rho}_0(t)|^{-3} + \frac{\boldsymbol{i}}{(V_1^2\tau^2)}\right] dt - \begin{pmatrix} a_0 \\ 4b_0 \end{pmatrix} \frac{\delta}{V_1^2} + O(\delta^2)$$

where

$$\boldsymbol{\rho}_0 = \boldsymbol{r}_0 - \boldsymbol{r}_m, \quad a_0 = (1 - 3\alpha_0^2)/6|\boldsymbol{r}_m(t_1)|^3$$

$$b_0 = -\alpha_0\beta_0/2|\boldsymbol{r}_m(t_1)|^3$$

and α_0 and β_0 are the direction cosines of $\boldsymbol{r}_m(t_1)$ in the $(\boldsymbol{i}, \boldsymbol{j})$ frame. The $O(\delta^3)$ terms in $\boldsymbol{\rho}_{1b}(t_1)$ and the $O(\delta^2)$ terms in $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ are determined below when $\boldsymbol{r}_m(t)$ describes a circular orbit.

From the above results, it follows that in order to obtain eight figure accuracy in the computation of $\boldsymbol{\rho}_{1b}(t_1)$, one could choose $\delta = O(10^{-3})$ and would then be subtracting quantities of $O(10^3)$ in the aforementioned integrand for $\boldsymbol{\rho}_{1b}(t_1)$ over $[t_0, t_1 - \delta]$. It would therefore be necessary to carry 12 digits in the computation of the $\boldsymbol{\rho}_{1b}$-integral over $[t_0, t_1 - \delta]$ (at least near $t_1 - \delta$). To compute $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ to eight figures is more difficult, and it would be easier to determine the $O(\delta^2)$ term in the above expression for $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ in which case the error in the integration over $[t_1 - \delta, t_1]$ would be of $O(\delta^3)$ and one could choose $\delta = O(10^{-3})$ and would be subtracting quantities of $O(10^6)$ in the aforementioned integrand for $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ over $[t_0, t_1 - \delta]$. It would therefore be necessary to carry 15 digits in the computation of the $\dot{\boldsymbol{\rho}}_{1b}$-integral over $[t_0, t_1 - \delta]$ (at least near $t_1 - \delta$).

A derivation of the preceding results for $\boldsymbol{\rho}_{1b}(t_1)$ and $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ is now given. It follows from the differential equations and initial conditions for $\boldsymbol{r}_m(t)$ and $\boldsymbol{r}_0(t)$, Eqs. (2) and (3), respectively, that in $(\boldsymbol{i}, \boldsymbol{j})$ coordinates

$$\boldsymbol{\rho}_0(t) = V_1\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\tau + \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}\tau^3 + \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}\tau^4 + O(\tau^5)\right]$$

where the constants a_0 and b_0 are defined above and where for $\boldsymbol{r}_m(t)$ describing a circular orbit

$$c_0 = [V_1\alpha_0/8|\boldsymbol{r}_m(t_1)|^4](3\beta_0^2 - 2\alpha_0^2) - [\alpha_0\beta_0/2|\boldsymbol{r}_m(t_1)|^{9/2}]$$

$$d_0 = [V_1\beta_0/8|\boldsymbol{r}_m(t_1)|^4](\beta_0^2 - 4\alpha_0^2) + [(\alpha_0^2 - \beta_0^2)/4|\boldsymbol{r}_m(t_1)|^{9/2}]$$

Thus, in $(\boldsymbol{i}, \boldsymbol{j})$ coordinates it follows that

$$\frac{-\boldsymbol{\rho}_0(t)}{|\boldsymbol{\rho}_0(t)|^3} = \frac{1}{V_1^2\tau^2}\begin{pmatrix} 1 + 2a_0\tau^2 + 2c_0\tau^3 + O(\tau^4) \\ -b_0\tau^2 - d_0\tau^3 + O(\tau^4) \end{pmatrix}$$

Next, from the differential equations and initial conditions satisfied by the transition matrix, Eq. (4), it follows that

$$\Phi_{rv}(t_1, t) = I\tau + \tfrac{1}{6}(d\boldsymbol{F}/d\boldsymbol{r})[\boldsymbol{r}_0(t_1)]\tau^3 - \tfrac{1}{12}(d^2\boldsymbol{F}/d\boldsymbol{r}^2)[\boldsymbol{r}_0(t_1)]\dot{\boldsymbol{r}}_0(t_1)\tau^4 + O(\tau^5)$$

and

$$\Phi_{vv}(t_1, t) = I + \tfrac{1}{2}(d\boldsymbol{F}/d\boldsymbol{r})[\boldsymbol{r}_0(t_1)]\tau^2 - \tfrac{1}{3}(d^2\boldsymbol{F}/d\boldsymbol{r}^2)[\boldsymbol{r}_0(t_1)]\dot{\boldsymbol{r}}_0(t_1)\tau^3 + O(\tau^4)$$

where in $(\boldsymbol{i}, \boldsymbol{j})$ coordinates it can be shown that

$$\tfrac{1}{6}(d\boldsymbol{F}/d\boldsymbol{r})[\boldsymbol{r}_0(t_1)] = -\begin{bmatrix} a_0 & b_0 \\ b_0 & l_0 \end{bmatrix}$$

where $l_0 = (3\alpha_0^2 - 2)/6|\boldsymbol{r}_m(t_1)|^3$ and

$$-\tfrac{1}{3}(d^2\boldsymbol{F}/d\boldsymbol{r}^2)[\boldsymbol{r}_0(t_1)]\dot{\boldsymbol{r}}_0(t_1) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

where, for $\boldsymbol{r}_m(t)$ describing a circular lunar orbit

$$m_{11} = [V_1\alpha_0/|\boldsymbol{r}_m(t_1)|^4](2\alpha_0^2 - 3\beta_0^2) + [2\alpha_0\beta_0/|\boldsymbol{r}_m(t_1)|^{9/2}]$$

$$m_{12} = m_{21} = [V_1\beta_0/|\boldsymbol{r}_m(t_1)|^4](4\alpha_0^2 - \beta_0^2) + [(\beta_0^2 - \alpha_0^2)/|\boldsymbol{r}_m(t_1)|^{9/2}]$$

$$m_{22} = [V_1\alpha_0/|\boldsymbol{r}_m(t_1)|^4](4\beta_0^2 - \alpha_0^2) - [2\alpha_0\beta_0/3|\boldsymbol{r}_m(t_1)|^{9/2}]$$

It then follows that

$$-\left[\Phi_{rv}(t_1, t)\frac{\boldsymbol{\rho}_0(t)}{|\boldsymbol{\rho}_0(t)|^3} + \frac{\boldsymbol{i}}{V_1^2\tau}\right] = \frac{1}{V_1^2}\left[\begin{pmatrix} a_0 \\ -2b_0 \end{pmatrix}\tau + \begin{pmatrix} 2c_0 + m_{11}/4 \\ -d_0 + m_{21}/4 \end{pmatrix}\tau^2 + O(\tau^3)\right]$$

and that

$$-\left[\Phi_{vv}(t_1, t)\frac{\boldsymbol{\rho}_0(t)}{|\boldsymbol{\rho}_0(t)|^3} + \frac{\boldsymbol{i}}{V_1^2\tau^2}\right] = \frac{1}{V_1^2}\left[-\begin{pmatrix} a_0 \\ 4b_0 \end{pmatrix} + \begin{pmatrix} 2c_0 + m_{11} \\ -d_0 + m_{21} \end{pmatrix}\tau + O(\tau^2)\right]$$

Finally, integrating these expressions from $t_1 - \delta$ to t_1 yields the above expressions for $\boldsymbol{\rho}_{1b}(t_1)$ and $\dot{\boldsymbol{\rho}}_{1b}(t_1)$ with the $O(\delta^3)$ term in $\boldsymbol{\rho}_{1b}(t_1)$ given by

$$\frac{1}{V_1^2}\begin{pmatrix} 2c_0 + m_{11}/4 \\ -d_0 + m_{21}/4 \end{pmatrix}\frac{\delta^3}{3}$$

and the $O(\delta^2)$ term in $\dot{\rho}_{1b}(t_1)$ given by

$$\frac{1}{V_1^2}\begin{pmatrix} 2c_0 + m_{11} \\ -d_0 + m_{21} \end{pmatrix}\frac{\delta^2}{2}$$

when $r_m(t)$ describes a circular orbit.

This completes the description of the computational technique used in first-order asymptotic matching to obtain the parameters of the moon centered hyperbola.

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Uncertainty in Measurement of Intermittency in Turbulent Free Shear Flows

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THE phenomenon of intermittency has come to be recognized as one of fundamental importance in turbulent free shear flows, although it has not been incorporated in turbulence theories to any extent. The intermittency factor (γ) has been measured in a variety of flows by various workers, e.g., Townsend in the wake of cylinders,[1] Corrsin and Kistler in boundary layers and round jets,[2] Bradbury in plane jets,[3] etc.

There are a number of methods of measuring γ. The direct method is to photograph the amplified hot-wire signal (either the u-component of turbulent fluctuations or its time derivative) and measure the duration of turbulent flow. The signal can be recorded on tape and analyzed by computer. An electronic circuit can be built which will indirectly measure the duration of turbulent bursts. This is the most common means of measuring γ. The purpose of this note is to focus attention on one of the inherent difficulties of using such a circuit, namely the calibration. This has been mentioned by Townsend[1] and by Corrsin and Kistler.[2] However, they have not given details regarding the extent of error that is involved. During the course of measurement of turbulent quantities in interacting wakes, the present authors have conducted a systematic analysis of this problem. The results are discussed below.

Figure 1 is a block diagram of the circuit that was used for measuring γ. The amplified, filtered, rectified output of the anemometer is lead into the Schmidt trigger. The output of the trigger is zero so long as the input is below a preset threshold value. This is the upper trip point (or UTP) when the input voltage exceeds the UTP the trigger output is a pulse of constant amplitude. This output continues so long as the input level is above a second trip point called the lower trip point (or LTP). When the input to the trigger falls below the LTP, the trigger output ceases. Because of the nature of the trigger design the two trip points cannot be made identical. However, they can be made sufficiently close that errors are not significant. This will be at the expense of the frequency response of the trigger. For present purposes this problem is not serious.

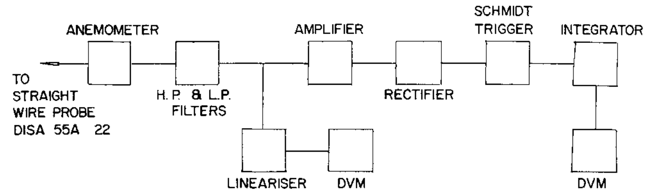

Fig. 1 Block diagram for measuring intermittency factor.

The main difficulty is in setting the threshold level or the UTP. This can be looked at from another point of view. Assuming the UTP has been fixed, to what level should the hot-wire signal be amplified? In the present investigation this view was adopted for the reason that it is possible to vary the amplification over a large range.

The hot-wire probe was located in the wake of a cylinder of 0.5 in. diam, at a downstream distance of 32.25 in. from the cylinder. The freestream velocity was held constant at about 87 fps. This gave a Reynolds number (based on the cylinder diameter) of 22,000. Keeping other factors constant, the amplifier gain was varied from about 150 to 320. At different values of gain, the integrator output was measured at three different locations (in the same plane). One point was along the wake symmetry axis ($y = 0$ in.), another at the edge of the wake ($y = 4$ in.) and the third in between ($y = 3.25$ in.).

The integrator output is shown plotted in Fig. 2 against the inverse of the amplification on semilog coordinates. It is seen that output varies exponentially with gain. The equation to these lines may be written as

$$\log E_{\text{int}} = (m/G) + \log C \tag{1}$$

where E_{int} is integrator output, G is amplification or gain, C is a constant and m is the slope of the lines. There are two points to be noted here: 1) C is nearly equal to the output of the integrator along the symmetry axis. This is to be expected since along the axis of the wake the flow is fully turbulent throughout the period of observation. Hence, the trigger would be fully open; 2) The slope of the lines, m, is a function of the intermittency factor at the location. This can be more clearly observed from Fig. 3. In this figure the integrator output at any point has been normalized

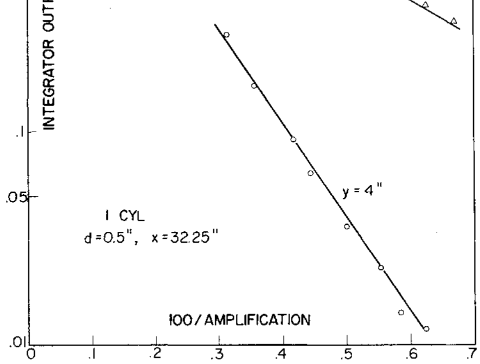

Fig. 2 Effect of amplification on measurement of intermittency factor (integrator output).

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